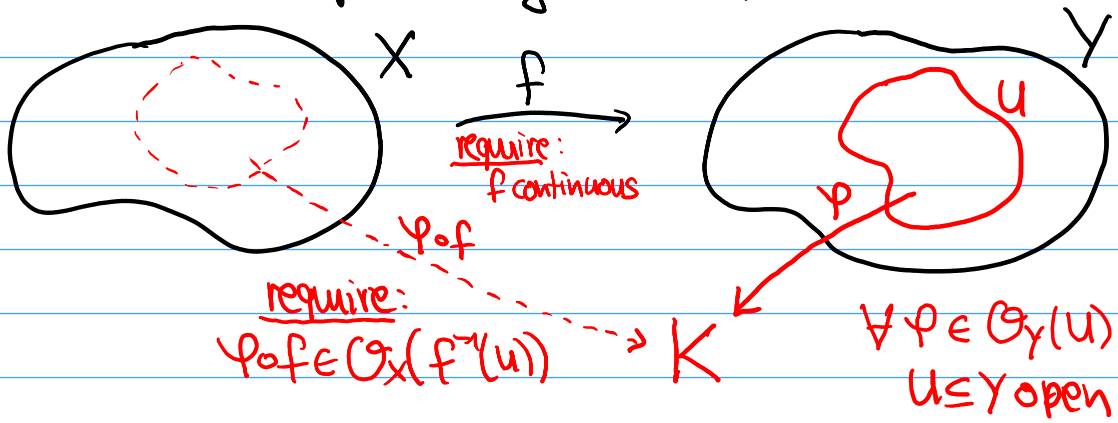
# 4. Morphisms

**Big picture**

→ have defined affine varieties $X$ and regular functions
$$X \overset{\text{open}}{\supseteq} U \longrightarrow \mathbb{A}^1$$
→ now: define arbitrary morphisms $X \to Y$ between aff. var.
→ Key idea: if we know what "nice" (e.g. regular) functions on $X, Y$ are, we recognize "nice maps" (e.g. morphisms) $X \to Y$ as those maps preserving these functions.




↳ This idea works for arbitrary top. spaces $X, Y$ with sheaves $\mathcal{O}_X, \mathcal{O}_Y$ of rings.

**Def** (Ringed spaces)
(a) A <u>ringed space</u> is a top. space $X$ together with a sheaf of rings on $X$ (denoted: $\mathcal{O}_X$, the <u>structure sheaf</u>)
Write: $(X, \mathcal{O}_X)$ or simply $X$
(b) $X$ affine variety $\rightsquigarrow$ always take $\mathcal{O}_X$ = regular fcts.
(c) An open subset $U \subseteq X$ is a ringed space with respect to the sheaf $\mathcal{O}_U = \mathcal{O}_X|_U$ restricted from $X$.

One issue for picture above:
Not clear in general what $\varphi \circ f \in \mathcal{O}_X(f^{-1}(U))$ means!
[$\varphi \in \mathcal{O}_Y(U)$ is some element of some ring]
Solution: restrict to sheaves of rings that are functions to $K$.

**Convention** From now to Chapter 12 (Schemes):
Assume for every ringed space $(X, \mathcal{F})$ that for $U \subseteq X$ open, we have

$$\mathcal{F}(U) \subseteq \{\varphi : U \to K : \text{U function}\}$$

pointwise addition & multiplication

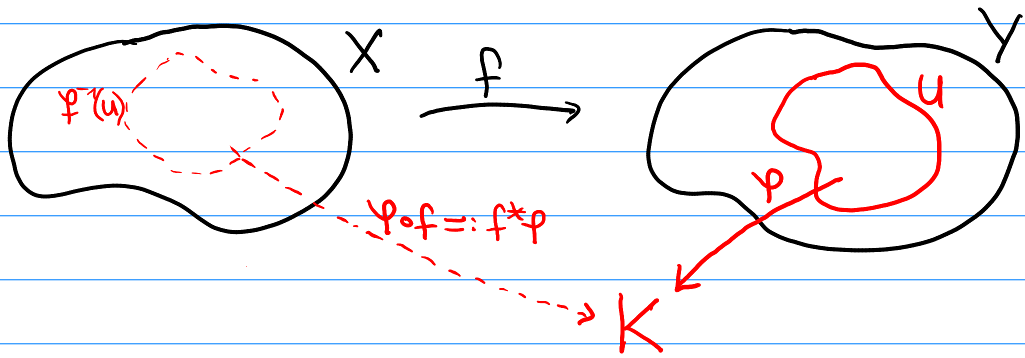
assume: • $\mathcal{F}(U)$ contains the constant fcts. to $K$.
• restriction maps of $\mathcal{F}$ = usual restrict. of fcts.

$$\boxed{\text{Summary: Every sheaf of rings is assumed to be a sheaf of } K\text{-valued functions.}}$$

assuming all sections are functions to K

using ringed spaces to define morphisms

**Def** (Morphisms of ringed spaces)
Let $f : X \longrightarrow Y$ be a map of ringed spaces.
(a) For any map $\varphi : U \to K$ from an open subset $U \subseteq Y$, we denote the composition $\varphi \circ f : f^{-1}(U) \longrightarrow K$ by $f^*\varphi$.
It is called the pull-back of $\varphi$ by $f$.



(b) The map $f$ is called a <u>morphism</u> (of ringed spaces) if it is continuous and for all open subsets $U \subseteq Y$ and $\varphi \in \mathcal{O}_Y(U)$ we have

$$f^*\varphi \in \mathcal{O}_X(f^{-1}(U)) \subseteq \{\psi : f^{-1}(U) \to K\}$$

Know what $f^*\varphi$ is as a function on $f^{-1}(U)$

$\Rightarrow$ pulling back by $f$ gives a $K$-algebra morphism

$$f^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$$
$$\varphi \longmapsto f^*\varphi.$$

(c) We say that $\varphi$ is an <u>isomorphism</u> (of ringed spaces) if it has a two-sided inverse, i.e. if it is bijective and both $f: X \to Y$ and $f^{-1}: Y \to X$ are morphisms.

In particular:
(Iso) morphisms of (open subsets of) affine varieties
= (Iso) morphisms as ringed spaces.

<u>Rmks</u>
(a) $f$ continuous ensures $f^{-1}(U) \subseteq X$ open for $U \subseteq Y$ open
(b) Without our Convention (sheaves of rings = sheaves of fcts.)
$\rightsquigarrow$ Data of morphism $f: X \to Y$ would have to include ring morphisms

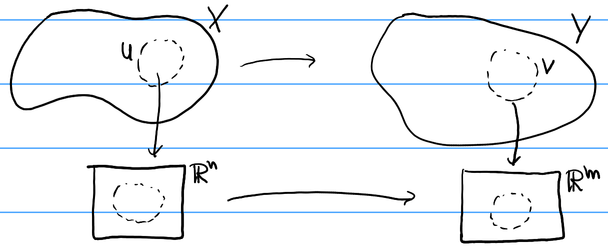
$$f^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$$

This we'll do later when discussing schemes.

## Exercise for differential geometers

(a) Given a smooth real manifold $X$, define the sheaf $\mathcal{O}_X^{sm}$ of smooth functions on $X$.

(b) Show that a map $X \xrightarrow{f} Y$ of such manifolds is smooth if and only if $(X, \mathcal{O}_X^{sm}) \xrightarrow{f} (Y, \mathcal{O}_Y^{sm})$ is a morphism.




## Properties of morphisms of ringed spaces

The following are some basic results on morphisms of ringed spaces $X, Y, \dots$ which follow from the definition

(a) The identity map $id_X : X \to X$ is a morphism.

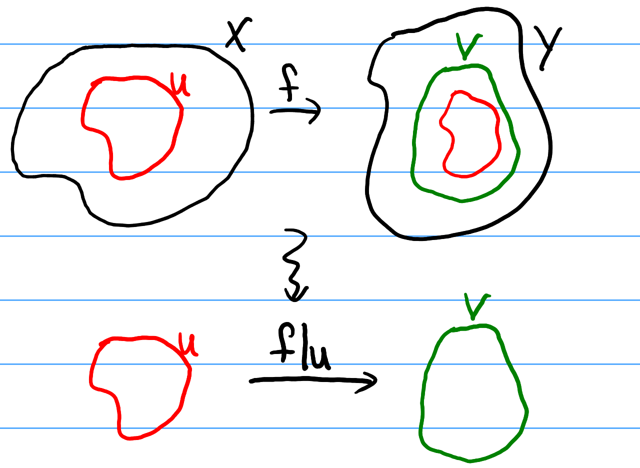
(b) For $f : X \to Y$ and $g : Y \to Z$ morphisms of ringed spaces, also the composition $g \circ f : X \to Z$ is a morphism.

make Ringed Spaces a Category!

(c) Restrictions of morphisms are morphisms:

If $f : X \to Y$ is a morphism of ringed spaces and $U \subseteq X$, $V \subseteq Y$ are open subsets such that

$$f(U) \subseteq V$$




then also $f|_U : U \to V$ is a morphism of ringed spaces.

Special example: the inclusion $U \hookrightarrow X$ of any open subset

**Subtle exercise** Show that the notion of isomorphism from the first defin. we gave is equivalent to the notion of isomorphism in the category RingedSpaces.

**Next** Instead of starting w/ morphism $X \to Y$ and restricting to open subsets of $X$, we can also start with an open cover $X = \bigcup_i U_i$ of $X$ and morphisms $U_i \longrightarrow Y$ and glue them together.

**Lem** Let $f: X \longrightarrow Y$ be a (set-theoretic) map of ringed spaces. Assume that there is an open cover $\{U_i : i \in I\}$ of $X$ such that all restrictions $f|_{U_i} : U_i \longrightarrow Y$ are morphisms. Then $f$ is a morphism.

**Proof** By def. of morphisms, we have to check two properties:

(a) $f$ is continuous: **Slogan** being contin. can be checked on an open cover.

Let $V \subseteq Y$ be open.

$$\Rightarrow f^{-1}(V) = \bigcup_{i \in I} (U_i \cap f^{-1}(V)) = \bigcup_{i \in I} \underbrace{(f|_{U_i})^{-1}(V)}_{\substack{\text{open in } U_i \text{ since } f|_{U_i} \text{ morph., hence continuous} \\ \Rightarrow \text{open in } X.}}$$

$\rightsquigarrow f^{-1}(V)$ open as union of opens.

(b) $f$ pulls back sections of $\mathcal{O}_Y$ to sections of $\mathcal{O}_X$:

Let $V \subseteq Y$ be open, $\varphi \in \mathcal{O}_Y(V)$.

Then

$$(f^* \varphi)\big|_{U_i \cap f^{-1}(V)} = (\underbrace{f|_{U_i \cap f^{-1}(V)}}_{\substack{\text{morphism as restrict. of} \\ \text{morphism } f|_{U_i}}})^* \varphi \in \mathcal{O}_X(U_i \cap f^{-1}(V))$$

$\mathcal{O}_X$ sheaf and on open cover $f^{-1}(V) = \bigcup_{i \in I} (U_i \cap f^{-1}(V))$ have sections $(f^*\varphi)|_{U_i \cap f^{-1}(V)}$ which agree on overlaps $U_i \cap U_j \cap f^{-1}(V)$.

$\Rightarrow$ by gluing property of sheaves there is some section $\psi \in \mathcal{O}_X(f^{-1}(V))$ restricting to $(f^*\varphi)|_{U_i \cap f^{-1}(V)}$

$\Rightarrow \psi = f^* \varphi$ is section of $\mathcal{O}_X$ on $f^{-1}(V)$. $\square$

## Morphisms between affine varieties

$\rightsquigarrow$ apply general concept of morph. of ringed spaces to (open subsets of) affine varieties.

<u>Pro</u> (Morphisms between affine varieties)
Let $U$ be an open subset of the affine variety $X$ and $Y \subseteq \mathbb{A}^n$ be another affine variety. Then the morphisms $f: U \to Y$ are exactly the maps of the form

$$f = (\varphi_1, \dots, \varphi_n) : U \to Y, \; x \mapsto (\varphi_1(x), \dots, \varphi_n(x)) \qquad (*)$$

with $\varphi_i \in \mathcal{O}_X(U)$ for all $i = 1, \dots, n$.
In particular: $\{\text{morphisms } U \to \mathbb{A}^1\} \triangleq \mathcal{O}_X(U)$.

<u>Pf</u> $\boxed{\text{Every morphism } f \text{ is of form } (*)}$

First assume $f: U \to Y$ is a morphism.
For $i = 1, \dots, n$, the $i$th coordinate function $Y \to K$, $y = (y_1, \dots, y_n) \mapsto y_i$ is a regular function on $Y$.

$\Rightarrow \underbrace{\varphi_i := f^* y_i}_{\varphi_i = i\text{-th coordinate of map } f} \in \mathcal{O}_X(f^{-1}(Y)) = \mathcal{O}_X(U)$ is regular by definition

$\Rightarrow f = (\varphi_1, \dots, \varphi_n)$ has desired form.

$\boxed{\text{Every map } f \text{ of the form } (*) \text{ is a morphism}}$

Let $f = (\varphi_1, \dots, \varphi_n): U \to \mathbb{A}^n$ with $\varphi_i \in \mathcal{O}_X(U)$ and $f(U) \subseteq Y$.

$\to f$ is continuous: $Z \subseteq Y$ closed $\rightsquigarrow Z = V_Y(g_1, \dots, g_r)$ w/ $g_i \in A(Y)$

$\Rightarrow f^{-1}(Z) = \{x \in U : g_i(\underbrace{\varphi_1(x), \dots, \varphi_n(x)}_{\in \mathcal{O}_X(U) \text{ since } \mathcal{O}_X(U) \text{ is } K\text{-algebra}}) = 0 \text{ for } i = 1, \dots, r\}$

$g_i$ = polynomial

<u>Exa</u> $g_1(y_1, y_2) = 3y_1^2 + y_1 \cdot y_2 + 5$, $\varphi_1(x) = \frac{x_1}{x_2}$, $\varphi_2(x) = \frac{1}{x_1+1}$

$\rightsquigarrow g_1(\varphi_1(x), \varphi_2(x)) = 3 \cdot \left(\frac{x_1}{x_2}\right)^2 + \left(\frac{x_1}{x_2}\right) \cdot \left(\frac{1}{x_1+1}\right) + 5 \rightsquigarrow$ still regular funct.

$$\Rightarrow f^{-1}(Z) = \underbrace{V(\underbrace{g_1 \circ f}_{\in \mathcal{O}_X(U)})}_{\text{closed in } U \text{ [Lem 3.4]}} \cap \dots \cap \underbrace{V(g_r \circ f)}_{\text{closed}}$$

closed in $U$.

$\rightarrow$ $f$ pulls back regular fcts. to regular functs.

$\psi \in \mathcal{O}_Y(W)$ regular function, $a \in f^{-1}(W)$, $y = f(x) \in W$

$\Rightarrow$ $\psi = \frac{g}{h}$ around $y$ for $g, h \in A(Y)$

$\Rightarrow$ $f^*\psi = \dfrac{g(\varphi_1(x), \dots, \varphi_n(x))}{h(\varphi_1(x), \dots, \varphi_n(x))}$ $\leftarrow \varphi_i =$ quotients of polynomials in $x = (x_1, x_2, \dots)$ around $x = a$

$\Rightarrow$ after expanding polynomials $g, h$, this becomes quotient of polynomials in $x$.

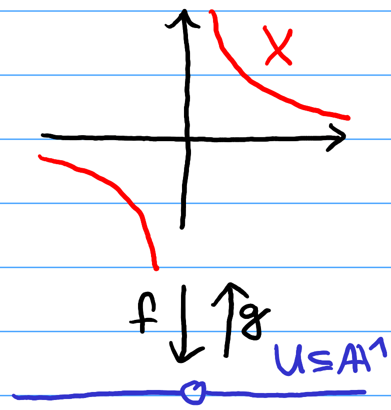
$\Rightarrow$ $f^*\psi \in \mathcal{O}_X(f^{-1}(W))$

$\Rightarrow$ $f$ is a morphism. $\square$

<u>Exa</u> $X = V(xy - 1) \subseteq \mathbb{A}^2$, $Y = \mathbb{A}^1$

$\Rightarrow$ $f : X \to Y$, $(x, y) \mapsto x$

is morphism.




$U = \mathbb{A}^1 \setminus \{0\} \subseteq \mathbb{A}^1$, then

$$g : U \to X, \quad x \mapsto (\underset{x \in \mathcal{O}_Y(U)}{x}, \underset{1/x \in \mathcal{O}_Y(U)}{1/x})$$

is also morphism.

We have: $g \circ f = \mathrm{id}_X$, $f \circ g = \mathrm{id}_U \Rightarrow (U, \mathcal{O}_U) \underset{f}{\overset{g}{\rightleftarrows}} (X, \mathcal{O}_X)$ are inverse isomorphisms!

<u>Note</u> Here we see that it was important to allow regular fcts. with denominators!

## Morphisms of affine varieties via coordinate rings

For affine varieties, their morphisms are entirely described by homomorph. of their coordinate rings (in opposite direction)

**Cor** For any two affine varieties $X, Y$, there is a bijection

$$\{\text{morphisms } X \to Y\} \xleftrightarrow{1:1} \left\{\begin{array}{c}K\text{-algebra homomorph.} \\ A(Y) \to A(X)\end{array}\right\}$$

$$f \longmapsto f^*.$$

**Pf** By def: $f^* : \underbrace{\mathcal{O}_Y(Y)}_{A(Y)} \longrightarrow \mathcal{O}_X(f^{-1}(Y)) = \underbrace{\mathcal{O}_X(X)}_{A(X)}$ ring hom.

**Conversely:** Let $g : A(Y) \to A(X)$ be ring hom.
Assume $Y \subseteq \mathbb{A}^n$ and let $y_1, \dots, y_n \in A(Y)$ be coord. funct.
$\rightsquigarrow \varphi_i = g(y_i) \in A(X)$ and we set

$$f = (\varphi_1, \dots, \varphi_n) : X \longrightarrow \mathbb{A}^n.$$

Then: for any $h \in K[y_1, \dots, y_n]$ we have

$$(\star) \quad (f^* h)(x) = h(f(x)) = h(\underbrace{\varphi_1(x)}_{g(y_1)(x)}, \dots, \underbrace{\varphi_n(x)}_{g(y_n)(x)}) = g(h(y_1, \dots, y_n))(x)$$

$\uparrow$ $g$ is $K$-algebra morphism $\rightsquigarrow$ commutes w/ $+, \cdot$, thus also w/ poly. $h$.

**Claim 1** $f(X) \subseteq Y$
Indeed: for $h \in I(Y) \rightsquigarrow f^* h = g(\underbrace{h}_{h=0 \text{ in } A(Y)}) = 0 \in A(X)$
$\Rightarrow \forall x \in X : f(x) \in V(h)$
$\Rightarrow f(X) \subseteq V(I(Y)) \overset{\text{Nullstellen-}}{\underset{\text{satz}}{=}} Y.$

**Claim 2** $f$ is a morphism. Indeed: coord. fcts. $\varphi_i \in \mathcal{O}_X(X)$. + Prop above

**Claim 3** $f^* = g \rightsquigarrow$ Yes by $(\star)$. $\square$

**Cor** $\{\text{isomorphisms } X \xrightarrow{\sim} Y\} \triangleq \{\text{isomorphisms } K(Y) \xrightarrow{\sim} K(X)\}$.

**Pf** $f: X \to Y$ isom w/ inverse $g: Y \to X$

$\Rightarrow f^*: K(Y) \to K(X)$ " " " $g^*: K(X) \to K(Y)$.

Indeed:

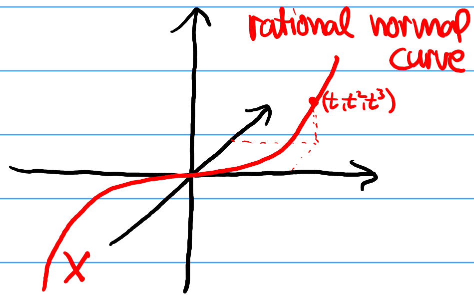
$$f^* \circ g^* = (g \circ f)^* = (\mathrm{id}_X)^* = \mathrm{id}_{K(X)} \text{ ; similar } g^* \circ f^* = \mathrm{id}_{K(Y)} \quad \square$$

**Exa** $X = V(y - x^2, z - x^3) \subseteq \mathbb{A}^3$

$\Rightarrow A(X) = K[x,y,z]/\langle y - x^2, z - x^3 \rangle$

$\cong K[x] = A(\mathbb{A}^1)$

$\Rightarrow X \cong \mathbb{A}^1$ via $(x,y,z) \mapsto x$.



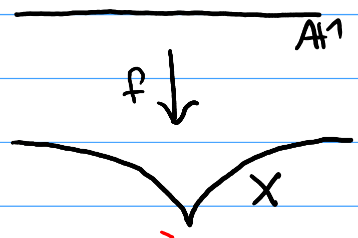

**Exercise** Show that the set of automorphisms

$$\mathrm{Aut}(\mathbb{A}^1) = \{f: \mathbb{A}^1 \to \mathbb{A}^1 : f \text{ isomorphism}\}$$

is given by the functions $f(x) = ax + b$, $a \in K^\times$, $b \in K$.

**Exa** (Isomorphisms $\neq$ bijective morphisms)

Let $X = V(x^2 - y^3) \subseteq \mathbb{A}^2$ be the cubic curve on the right




$$f: \mathbb{A}^1 \to X, \quad t \mapsto (t^3, t^2)$$

(0,0) is "singular point" in sense defined in Ch. 10

is a morphism which is bijective with (set-theoretic) inverse map:

$$f^{-1}: X \to \mathbb{A}^1, \quad (x,y) \mapsto \begin{cases} x/y & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}.$$

But: $f$ is not an isomorphism (i.e. $f^{-1}$ is not a morphism)!

$$\rightsquigarrow f^*: A(X) = K[x,y]/\langle x^2 - y^3 \rangle \longrightarrow K[t]$$

not an iso.

$$[x] \longmapsto t^3$$
$$[y] \longmapsto t^2$$

$t \in K[t]$ not in image of $f^*$.

## Products of affine varieties revisited

Have seen $X, Y$ affine varieties $\Rightarrow X \times Y$ aff. var., $A(X\times Y) = A(X) \otimes A(Y)$

Now: give abstract characterization of product $X \times Y$

(Later: use such property to define products of abstract varieties)

**Preparation** $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ aff. varieties

$\Rightarrow \pi_X : X \times Y \longrightarrow X$, $(x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (x_1, \ldots, x_n)$

is a morphism (components = regular fcts. on $X \times Y$)

similar: $\pi_Y : X \times Y \longrightarrow Y$.

**Pro** (Universal property of products)

Let $X, Y$ be affine varieties and let $\pi_X, \pi_Y$ be the projections from $X \times Y$ to $X, Y$, respectively. Then for any affine variety $Z$ and morphisms $f_X : Z \to X$, $f_Y : Z \to Y$ there is a unique morphism $f : Z \longrightarrow X \times Y$ such that

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists! f"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\
& X \times Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \\
& Y &
\end{tikzcd}$$

$$f_X = \pi_X \circ f \quad \text{and} \quad f_Y = \pi_Y \circ f. \qquad (+)$$

**Slogan** Giving a morphism to $X \times Y$ is the same as giving a morphism to $X$ and to $Y$.

$$\rightsquigarrow \quad \mathrm{Mor}(Z, X \times Y) \xrightarrow[\text{bijection}]{\sim} \mathrm{Mor}(Z, X) \times \mathrm{Mor}(Z, Y)$$
$$f \longmapsto (\pi_X \circ f, \ \pi_Y \circ f)$$

**Pf** $f : Z \longrightarrow X \times Y$, $z \mapsto (f_X(z), f_Y(z))$ is only set-theoretic map w/ $(+)$

This is a morphism: all coordinates of maps $f_X, f_Y$ given by regular functions $\Rightarrow$ same is true for $f$. $\square$

## Interpretation in terms of coordinate rings

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists! f"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\ & X \times Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \\ & Y & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A(Z) & & \\ & A(X) \otimes A(Y) \arrow[ul, dashed, "\exists! g"'] & A(X) \arrow[l, "\pi_X^*"'] \arrow[ull, bend right, "g_X"'] \\ & A(Y) \arrow[u, "\pi_Y^*"] \arrow[uul, bend left, "g_Y"] & \end{tikzcd}$$

For all K-algebra homomorphism

$g_X : A(X) \longrightarrow A(Z)$

$g_Y : A(Y) \longrightarrow A(Z)$

there exists a unique K-alg. hom.

$g : A(X) \otimes A(Y) \longrightarrow A(Z)$

Satisfying $g \circ \pi_X^* = g_X$, $g \circ \pi_Y^* = g_Y$.

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{universal property of tensor products of K-algebras}}$

universal property of tensor products of K-algebras

**Note**

$\pi_X^*(\alpha) = \alpha \otimes 1 \in A(X) \otimes A(Y)$

## Correspondence between aff. varieties & fin. generated reduced K-algebras

Have seen: $X \subseteq \mathbb{A}^n$ affine variety

$\Rightarrow A(X) = K[x_1,\dots,x_n] / \underbrace{I(X)}_{\text{radical ideal}} \Rightarrow A(X)$ is finitely generated $\underbrace{\text{reduced}}_{0 \text{ is only nilpotent element}}$ K-algebra

Now, we present the opposite direction:

**Construction** Let R be a f.g. reduced K-algebra

Pick generators $b_1,\dots,b_n$ of R as K-algebra

$\Rightarrow$ obtain a surjective K-algebra homomorphism

$$g: K[x_1,\dots,x_n] \longrightarrow R, \; f \longmapsto f(b_1,\dots,b_n).$$

Isomorphism theorem: $R \cong K[x_1,\dots,x_n]/J$ with $J = \mathrm{Ker}(g)$.

R reduced $\Rightarrow$ J is radical ideal

$\rightsquigarrow X = V(J)$ is affine variety in $\mathbb{A}^n$ with $A(X) \cong R$.

Construction depended on choice of generators $b_1,\dots,b_n \in R$

Different choice $\rightsquigarrow X' \subseteq \mathbb{A}^m \leftarrow$ can choose diff. number of gens.

But then:

$$A(X) \cong R \cong A(X') \xRightarrow{\text{Cor}} X \cong X' \text{ (as ringed spaces)}$$

$\rightsquigarrow$ give slight re-definition of "affine variety" to be more intrinsic

**Def** From now on, an <u>affine variety</u> will be a ringed space that is isomorphic to an affine variety $X \subseteq \mathbb{A}^n$ in the old sense. $\rightsquigarrow$ topology, reg. functions, $A(X) = \mathcal{O}_X(X)$ still make sense!

**Note** Could call $X \subseteq \mathbb{A}^n$ an <u>embedded affine variety</u>, and $(X, \mathcal{O}_X)$ an <u>abstract affine variety</u>.

For simplicity, we'll just keep using "affine variety".

Illustration

embedded affine variety

abstract affine variety

$\mathbb{A}^n$ $X$ $(X, \mathcal{O}_X)$

## Categorical interpretation (aside)

Can summarize the above as a correspondence of categories:

→ Affine Varieties:

- objects: affine varieties $X = (X, \mathcal{O}_X)$ new sense
- Morphisms: $X \to Y$ morphism (of ringed spaces)

→ Finitely Generated Reduced K-algebras

- Objects: $R$ f.g. red. K-alg.
- Morphisms: $R \to S$ K-alg. morphism

Then:

$$\text{Affine Varieties} \xrightarrow{F} \text{Finitely Generated Reduced K-algebras}$$
$$X \longmapsto R = \mathcal{O}_X(X)$$
$$(X \xrightarrow{f} Y) \longmapsto (\mathcal{O}_X(X) \xleftarrow{f^*} \mathcal{O}_Y(Y))$$

is a contravariant functor that is an equivalence of categories

arrows change direction

↳ bijective on equiv. classes of objects

↳ $\mathrm{Mor}(X,Y) \cong \mathrm{Mor}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$

Later Build geometric objects (affine schemes) corresponding to more general rings.

## Open subsets of affine varieties

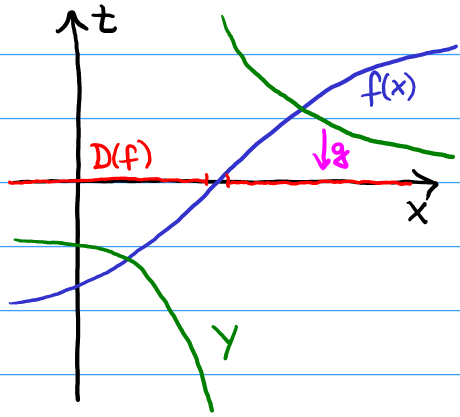
Some open subsets $U \subseteq X$ of aff. varieties are aff. varieties themselves

**Pro** (Distinguished open subsets are affine varieties)
Let $X$ be an affine variety and let $f \in A(X)$. Then the distinguished open subset $D(f)$ is an affine variety with coordinate ring $A(D(f)) \cong A(X)_f$.

**Proof**: We have

$$Y = \{(x,t) \in X \times \mathbb{A}^1 : t \cdot f(x) = 1\} \subseteq X \times \mathbb{A}^1$$

is an affine variety as the vanishing locus of the polynomial $t \cdot f(x) - 1 \in A(X \times \mathbb{A}^1)$ of the aff. var. $X \times \mathbb{A}^1$.
Have projection morphism




$$g : Y \longrightarrow D(f)$$
$$(x,t) \longmapsto x$$

with inverse

$$g^{-1} : D(f) \longrightarrow Y$$
$$x \longmapsto \Big(x, \underbrace{\tfrac{1}{f(x)}}_{\in \mathcal{O}_X(D(f))}\Big)$$

$\Rightarrow$ $D(f)$ is an affine variety and we have seen earlier

$$A(D(f)) \cong \mathcal{O}_X(D(f)) \underset{\text{Pro}}{\cong} A(X)_f. \qquad \square$$

However, not all open subsets of aff. varieties are aff. varieties:

Exa ($\mathbb{A}^2 \setminus \{0\}$ is not an affine var.)
We have seen for $U = \mathbb{A}^2 \setminus \{0\} \subseteq \mathbb{A}^2$ that

$$\mathcal{O}_U(U) \cong \mathcal{O}_{\mathbb{A}^2}(U) \cong K[x_1, x_2] \cong \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \qquad (**)$$

If $U$ was an affine variety $\overset{\text{Cor}}{\Longrightarrow}$ $(**)$ implies $U \cong \mathbb{A}^2$
with morphism given by
the identity $\lightning$ contradict.

the inclusion
$\mathrm{id}: U \longrightarrow \mathbb{A}^2$
satisfies $\mathrm{id}^*: \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \xrightarrow{\sim} \mathcal{O}_U(U)$
$\Rightarrow$ id must be isom. $\lightning$

However: $U$ is covered by dist. opens $\underbrace{D(x_1), D(x_2)}_{\text{affine varieties}} \subseteq \mathbb{A}^2$

Next chapter
Look more generally at ringed spaces which have covers by affine varieties.